# Restricted Three-Body Simplified Explicit Guidance Scheme

ROBERT C. STOW*
*Singer-General Precision Inc., Wayne, N. J.*

**This paper presents an analytical analysis of a first-order restricted three-body interplanetary guidance scheme using a vehicle and two finite mass gravitational bodies as the mathematical model. Analytical development utilizes a variation of orbital elements method in accounting for the perturbing effect of the secondary gravitational body. Both series and polynomial fitting approximations are used. The results indicate substantial increases in final position accuracy over a similar two body-analytical scheme, where only the primary gravitating body is considered. A target region in the proximity of the sphere of influence of the target body provides the area most applicable to the technique developed. Midcourse fixed time of arrival velocity correcting is demonstrated. The scheme is independent of any precomputed reference trajectory and of numerical integration and therefore allows application of its flexibility to a number of guidance techniques already developed.**

## Nomenclature

| | |
|---|---|
| $B(t_D,t_0)$ | = linear guidance correction matrix |
| $d$ | = distance from spacecraft to target planet |
| $K_M$ | = gravitational constant of the target planet |
| $K_s$ | = gravitational constant of the sun |
| $p$ | = distance from target planet to the sun |
| $\mathbf{P}$ | = target planet position vector |
| $r$ | = distance from spacecraft to the sun |
| $\mathbf{R}$ | = spacecraft position vector |
| $\mathbf{R}_D$ | = spacecraft position vector at $t_D$ before correction |
| $\mathbf{R}_{DT}$ | = desired spacecraft position vector at $t_D$ |
| $t$ | = time from launch |
| $t_D$ | = target point arrival time |
| $t_0$ | = time of desired correction |
| $\mathbf{V}$ | = spacecraft velocity vector |
| $V_x,V_y,V_z$ | = spacecraft velocity coordinates |
| $X,Y,Z$ | = spacecraft position coordinates |
| $\alpha_1,\alpha_2,\alpha_3$ | = spacecraft position coordinate at $t_0$ |
| $\alpha_4,\alpha_5,\alpha_6$ | = spacecraft velocity coordinates at $t_0$ |
| $\delta\mathbf{R}_D$ | = target position error vector before correction |
| $\delta\mathbf{V}_0$ | = vector velocity correction to be applied at $t_0$ |
| $(\dot{\ })$ | = $d(\ )/dt$ |
| $(\ )'_i$ | = $\partial(\ )/\partial\alpha_i \quad (i = 1,2,\ldots,6)$ |
| $(\ )_0$ | = variable at time $t_0$ |

## Introduction

A BASIC guidance problem faced in interplanetary flight once the navigation problem has been solved is the determination of the velocity correction needed at a particular time during flight to achieve the mission objectives. Many times these objectives are specified as reaching a target point (usually near some destination planet) at a given time. Such fixed-time-of-arrival guidance is the basic approach investigated in this paper. For guidance computation, one can consider either ground-based techniques, which are not limited by computational capabilities but must rely on a communication link with the spacecraft, or onboard methods, which are limited by computer weight and size.

Onboard guidance schemes have branched into two categories. One, called implicit guidance, relies on a precomputed reference trajectory[1] about which the spacecraft action is linearized. A series of perturbation matrices may be determined numerically before flight at various time points along the reference trajectory. These matrices may be used in determining the required velocity correction at a particular time, given a small deviation in position and velocity of the spacecraft; the on-board computation required here involves only matrix multiplication. Other implicit techniques not requiring matrix multiplication involve position and velocity offset methods. In position offsetting, a destination point position displacement is used to account for the integrated effect of the disturbing acceleration causing differences between the actual many-body trajectory and a two-body osculating conic, both originating at the correction point. Velocity offset techniques, on the other hand, utilize the difference between the many-body velocity and the two-body velocity required to reach a desired target point in a given elapsed time. In both methods onboard computation is required for solving Lambert's equation together with stored data to determine a velocity correction. The regions of applicability of these techniques, however, are limited to small deviations from the predicted reference. Thus, these methods are not applicable if a situation arises during flight where a sizeable course change is required.

The second category, called explicit guidance techniques, does not rely on precomputed reference trajectories, but for high accuracy does require onboard numerical integration of equations of motion. Position and velocity off-setting methods similar to those previously described are two of the schemes applicable here.[1] In addition to numerical integration, computational complexity is involved in performing onboard iterative solutions to a two-point boundary value problem.

In the foregoing developments the gravitational influences of many bodies are taken into account. This paper is concerned with a basically explicit technique which does not require numerical integration but is designed to give better accuracies than a strictly two-body Keplerian analysis. A physical model of restricted three-body motion is utilized, and results are obtained analytically. A guidance scheme has been formulated previous to this paper in which restricted three-body dynamics were used.[4] This analysis resulted in a series of graphs prepared before the flight based on the target planet of the mission and the sun. Knowing the position and velocity of the spacecraft at a particular time and the time of flight, these graphs could be used to find the equivalent central body location and mass value which, when used with the two-body equations of motion, will give the final position and velocity obtained in restricted three-body motion. Thus, a single virtual mass center was used to replace equivalently the effects of the two gravitating masses in the restricted three-body motion.[2] This in turn was used

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* Project Engineer, Analysis and Software Design Guidance Section. Associate AIAA.

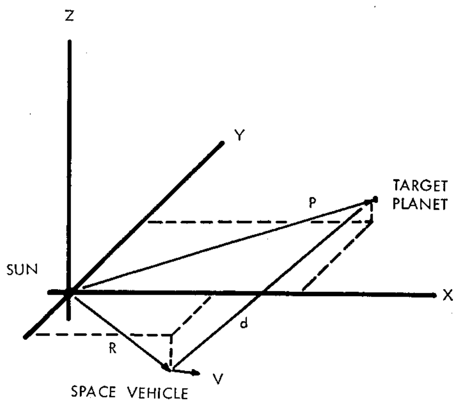


**Fig. 1 Restricted three-body motion in fixed coordinates.**

in a scheme to find the velocity correction. Thus, this technique seems to classify as an implicit technique relying on precomputed information to determine the correction. This paper presents a restricted three-body guidance scheme which does not depend on any precomputed information which would limit the flexibility of the technique. Analysis performed here as well as all the methods described above rely on the assumption of impulsive velocity correction of a coasting vehicle. Therefore, they are not necessarily valid for low thrusting over a sizable time interval.

## Analysis of Restricted Three-Body Motion

### System Mathematical Model

Restricted three-body motion provides the system mathematical model consisting of two finite gravitating bodies revolving in orbits around their common center of mass, and of an infinitesimal body (spacecraft) subject to their attraction. Gravitational effects of the two finite bodies are assumed to be caused only by their total masses concentrated at their respective mass centers. The primary finite body is taken as the sun and the other finite body is chosen to be the target planet of the spacecraft. Equations of motion for the spacecraft in this system may be derived in a fixed coordinate frame as indicated in Fig. 1, where $\mathbf{R}$ is the vector position of the spacecraft.[5]

$$\ddot{\mathbf{R}} + K_s(\mathbf{R}/r^3) = \nabla U \tag{1}$$

where

$$\Delta = \partial/\partial\mathbf{R},\ r = (\mathbf{R}\cdot\mathbf{R})^{1/2}$$

The potential function $U$ indicates the gravitational effect of the target planet on the spacecraft

$$U = K_M[1/d - (\mathbf{R}\cdot\mathbf{P})/p^3)] \tag{2}$$

where $d = [(\mathbf{R} - \mathbf{P})\cdot(\mathbf{R} - \mathbf{P})]^{1/2}$ represents the distance of the spacecraft from the target planet, and $p = (\mathbf{P}\cdot\mathbf{P})^{1/2}$ represents the target planet distance from the sun.

For the case where more than two gravitating bodies were to be considered, $U$ would represent the gravitational effects of all bodies other than the primary one and thus could mathematically be represented by a summation of terms similar to the one indicated previously.

### Variation of Orbital Parameters

From Eq. (1) describing the system mathematical model, the well-known two-body equations representing the relative dynamics of the spacecraft due to only the sun's gravity field are

$$\ddot{\mathbf{R}} + K_s(\mathbf{R}/r^3) = 0 \tag{3}$$

The analytical solution of this equation, which is the homogenous form of Eq. (1), may be determined in terms of six orbital constants of motion. Variation of orbital parameters utilizes the analytical flexibility of the two-body solution in conjunction with knowledge of the forcing function derived from the third body.[5] According to this conception, the spacecraft subject to the law of gravitation is always moving in a conic section, but in one which changes at each instant. The variable conic is tangent to the actual orbit at every point of it and thus is said to osculate with the actual orbit at the point of contact.

In astronautics the six constants of two-body motion of major interest are the three initial position and three initial velocity coordinates at the time of desired velocity correction $t_0$. Thus, the two-body solution used herein is the universal formulation of Kepler's equations developed in Ref. 5. Expressing this solution in vector form results in

$$\mathbf{R} = F\mathbf{R}_0 + G\mathbf{V}_0 \tag{4}$$

$$\mathbf{V} = F_1\mathbf{R}_0 + G_1\mathbf{V}_0 \tag{5}$$

where $F$, $G$, $F_1$ and $G_1$ are defined in Appendix A. In the aforementioned formulation $\mathbf{R}_0$ and $\mathbf{V}_0$ are vectors representing the six initial condition coordinates of the spacecraft at time $t_0$, and $\mathbf{R}$ and $\mathbf{V}$ are position and velocity vectors of the craft at time $t$. This solution is simultaneously valid for elliptic, parabolic, and hyperbolic two-body orbits.

Equation (1) defining the restricted three-body dynamics may be expressed in the following form:

$$\dot{\mathbf{R}} - \mathbf{V} = 0 \quad \dot{\mathbf{V}} + K_s(\mathbf{R}/r^3) = \nabla U \tag{6}$$

Assigning $\alpha_1,\alpha_2,\alpha_3$ to represent, respectively, the three initial position components of $\mathbf{R}_0$ and $\alpha_4,\alpha_5,\alpha_6$ to represent the three initial velocity components of $\mathbf{V}_0$, the position and velocity coordinates of the craft as indicated by the two-body solution may all be considered as functions of only $\alpha_1,\ldots,\alpha_6$ and $t - t_0$.

Considering the orbital elements $\alpha_i$ to vary with time, Eq. (6) may be written in terms of these elements through the two-body functional relationships described above in attempting to express the three-body problem with two-body-structured equations. Thus

$$\frac{\partial\mathbf{R}}{\partial t} - \mathbf{V} + \sum_{i=1}^{6}\left(\frac{\partial\mathbf{R}}{\partial\alpha_i}\right)\dot{\alpha}_i = 0 \tag{7}$$

$$\frac{\partial\mathbf{V}}{\partial t} + \frac{K_s\mathbf{R}}{r^3} + \sum_{i=1}^{6}\left(\frac{\partial\mathbf{V}}{\partial\alpha_i}\right)\dot{\alpha}_i = \nabla U \tag{8}$$

Equations (7) and (8) are greatly simplified from the fact that when $\alpha_1,\ldots,\alpha_6$ are constant the two-body equations represent the solution to Eq. (6) with $U = 0$.

The partial derivations $\partial\mathbf{R}/\partial t$ and $\partial\mathbf{V}/\partial t$, when $\alpha_1,\ldots,\alpha_6$ are regarded as variables, are identical with $\dot{\mathbf{R}}$ and $\dot{\mathbf{V}}$ when they are regarded as constants. Therefore the sum of the first two terms in each of Eqs. (7) and (8) are equal to zero leaving the following result,

$$\sum_{i=1}^{6}\frac{\partial\mathbf{R}}{\partial\alpha_i}\dot{\alpha}_i = 0,\quad \sum_{i=1}^{6}\left(\frac{\partial\mathbf{V}}{\partial\alpha_i}\right)\dot{\alpha}_i = \nabla U \tag{9}$$

By taking the dot product of these two vector equations with $\partial\mathbf{V}/\partial\alpha_K$ and $\partial\mathbf{R}/\partial\alpha_K$, respectively, the following equation results:

$$\sum_{i=1}^{6}[\alpha_K,\alpha_i]\dot{\alpha}_i = \nabla U\cdot\partial\mathbf{R}/\partial\alpha_K \qquad K = 1,\ldots,6 \tag{10}$$

where

$$[\alpha_K,\alpha_i] \equiv \left(\frac{\partial\mathbf{R}}{\partial\alpha_K}\right)\cdot\frac{\partial\mathbf{V}}{\partial\alpha_i} - \left(\frac{\partial\mathbf{V}}{\partial\alpha_K}\right)\cdot\frac{\partial\mathbf{R}}{\partial\alpha_i}$$

which are known as Lagrange's brackets.

One of the properties of Lagrange's brackets is that they do not contain time explicitly,[6]

$$\partial[\alpha_K,\alpha_i]/\partial t = 0 \tag{11}$$

Thus the brackets may be computed for any epoch, and in particular for $t = t_0$. Since the orbital elements $\alpha_i$ are the position and velocity coordinates at $t_0$, the easy computation of the brackets allows the simplification of Eq. (10) to the following form:

$$\dot{\alpha}_i = -\nabla U \cdot \partial \mathbf{R}/\partial \alpha_{i+3}, \qquad i = 1,2,3 \tag{12}$$

$$\dot{\alpha}_j = \nabla U \cdot \partial \mathbf{R}/\partial \alpha_{j-3}, \qquad j = 4,5,6 \tag{13}$$

Equations (12) and (13) are the differential equations summarizing the time variations of the two-body orbital elements required to describe the three-body-problem dynamics. Partial derivations of $U$ with respect to the components of $\mathbf{R}$ may be found from Eq. (2) whereas partial derivations of $\mathbf{R}$ with respect to $\alpha_1,\ldots,\alpha_6$ may be found from the two-body Eqs. (4) and (5). (Appendix A). Note that up to Eqs. (12) and (13) the development is equally applicable to a general potential function $U$ representing the effects of any number of additional gravitating bodies.

## Approximations in Variational Equation Solution

At this point, if numerical integration of Eqs. (12) and (13) is to be avoided, some approximations must be introduced. We may expand $\alpha_i$ as a power series in the target planet gravitational constant $K_M$

$$\alpha_i = \sum_{N=0}^{\infty} \alpha_{iN} K_M{}^N = \alpha_{i0} + \alpha_{i1}K_M + \alpha_{i2}K_M{}^2 + \ldots \tag{14}$$

This series will converge for some finite range of values of time greater than the flight times considered in this paper. These convergence properties are discussed in Ref. 4.

Equations (12) and (13) may be expanded in Taylor series about the values $\alpha_0$(i.e. $\alpha_{10},\alpha_{20},\ldots,\alpha_{60}$).

$$\dot{\alpha}_i = \dot{\alpha}_i\Big|_{\alpha=\alpha_0} + \sum_{K=1}^{6}\left(\frac{\partial \dot{\alpha}_i}{\partial \alpha_K}\Big|_{\alpha=\alpha_0}\right)(\alpha_K - \alpha_{K0}) + \ldots \tag{15}$$

$$i = 1,\ldots,6$$

Substituting Eq. (14) into Eq. (15) yields;

$$\dot{\alpha}_i = \dot{\alpha}_i|_{\alpha=\alpha_0} + \sum_{K=1}^{6} (\partial\dot{\alpha}_i/\partial\alpha_K|_{\alpha=\alpha_0})\, \alpha_{K1}K_M + \ldots \tag{16}$$

$$i = 1,\ldots,6$$

Note that in these equations the indicated partial derivatives are evaluated at the operating point $\alpha_0$. From Eqs. (2, 12, and 13) the target planet gravity constant $K_M$ may be factored out of each of the $\dot{\alpha}_i$ equations. Making this change in Eq. (16) and equating this result to the time derivative of Eq. (14) yields the following equalities between coefficients of like powers in $K_M$ up to second order,

$$\dot{\alpha}_{i0} = 0 \tag{17}$$

$$\dot{\alpha}_{i1} = K_M{}^{-1}\dot{\alpha}_i|_{\alpha=\alpha_0} \tag{18}$$

$$\dot{\alpha}_{i2} = K_M{}^{-1}\sum_{K=1}^{6}\left(\frac{\partial\dot{\alpha}_i}{\partial\alpha_K}\Big|_{\alpha=\alpha_0}\right)\alpha_{K1} \qquad i = 1,\ldots,6 \tag{19}$$

Solving Eq. (17) results in the zero-order coefficients of the assumed series expansion being constant in time. Thus, these six constants are the actual initial state coordinates of the spacecraft at time $t_0$. With these, Eq. (18) may be integrated and the terms of first-order are determined. Proceeding in the same manner, higher-order terms can be found. Analysis in this paper deals with only the zero- and first-order terms. If $\dot{\alpha}_i$ is small, the higher order terms may be neglected in the analysis. From a physical standpoint this is equivalent to requiring that the orbital elements in the variational analysis vary slowly with time.

To allow the first-order terms $\dot{\alpha}_{i1}$ to be determined analytically, the $\dot{\alpha}_{i1}$ functions are approximated by polynomials in time determined such that the polynomials yield the exact value of the respective functions at $n + 1$ selected time points $(t_0,t_1,t_2,\ldots,t_n)$.

$$\begin{aligned}\dot{\alpha}_{i1}(t) \approx\ & \frac{(t-t_1)(t-t_2)\cdots(t-t_n)}{(t_0-t_1)(t_0-t_2)\cdots(t_0-t_n)}\,\dot{\alpha}_{i1}(t_0) + \\ & \frac{(t-t_0)(t-t_2)(t-t_3)\cdots(t-t_n)}{(t_1-t_0)(t_1-t_2)(t_1-t_3)\cdots(t_1-t_n)}\,\dot{\alpha}_{i1}(t_1) + \\ & \ldots\frac{(t-t_0)(t-t_1)\cdots(t-t_{n-1})}{(t_n-t_0)(t_n-t_1)\cdots(t_n-t_{n-1})}\,\dot{\alpha}_{i1}(t_n)\end{aligned} \tag{20}$$

In this polynomial form the six first-order differential Eqs. (18) are easily analytically integrated from $t_0$ to $t_n$ to give the six first-order terms at time $t_n$. Since $\dot{\alpha}_{i1}$ varies slowly with time a fairly low-order polynomial may be realistically used. Care must be taken here in any attempt to try to improve accuracy by going to a very high-order polynomial fit since the coefficients indicated previously ultimately diverge. Other types of curve fits could be used here such as a repeated use of Simpson's rule utilizing a parabola fitting effectively three points at a time.

Thus, for a given flight time the perturbed elements appear to first-order as follows from Eq. (14):

$$\alpha_i = \alpha_{i0} + \alpha_{i1}K_M \qquad i = 1,\ldots,6 \tag{21}$$

These orbital elements may then be used in the two-body Kepler Eqs. (4) and (5) to determine the position and velocity of the craft at the desired time of arrival. Note that ephemerous data for the target planet is required to evaluate the partial derivatives of the potential function $U$ at the times required.

In addition, ephemerous data for all the planets could be utilized to evaluate partial derivatives of a more inclusive $U$. The same basic sets of derived equations would apply.

## Applications to Fixed Time of Arrival Guidance

If the aforementioned computed restricted three-body trajectory final position coordinates differ from the target point position on a proposed mission, this development can be incorporated into a guidance scheme. This scheme's purpose would be to determine the impulsive velocity correction required at time $t_0$ to reach a specified target point at a desired time of arrival. Many of the schemes presented in the introduction could be applied by essentially substituting the restricted three-body trajectory for the many-body integrated trajectory presently envisioned in these techniques.

The scheme investigated in this paper is a three-body linear guidance technique. Consider the situation presented in Fig. 2 where a vehicle is positioned at $D_1(t_0)$ with position vector $\mathbf{R}_0$ and velocity vector $\mathbf{V}_0$ with respect to the sun at time $t_0$. Desired vehicle position at time $t_D$ is $\mathbf{R}_{DT}$. Spacecraft position $\mathbf{R}_D$ at time $t_D$ if no velocity correction is made is easily found considering a three-body analysis to first order as just described. Thus, the target position error of the vehicle is

$$\delta\mathbf{R}_D = \mathbf{R}_{DT} - \mathbf{R}_D \tag{22}$$

The linear three-body guidance equation used to determine the required velocity correction at time $t_0$ to reduce the target position error $\delta\mathbf{R}_D$ to zero may be expressed as follows:

$$\delta\mathbf{V}_0 = B(t_D,t_0)^{-1}\delta\mathbf{R}_D \tag{23}$$

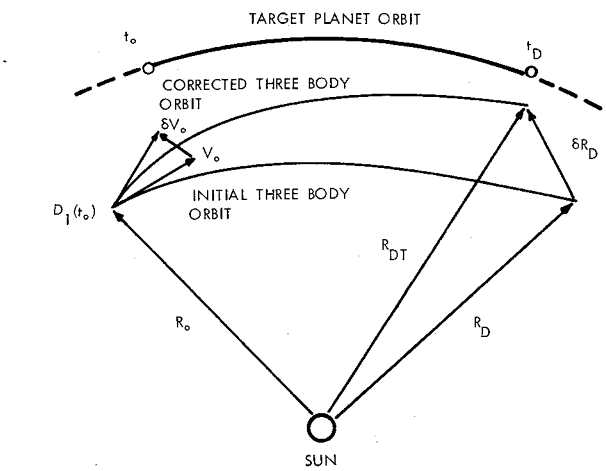


**Fig. 2 Fixed time of arrival linear guidance geometry.**

where

$$B(t_D,t_0) = \begin{bmatrix} \partial X_D/\partial V_{x0} & \partial X_D/\partial V_{y0} & \partial X_D/\partial V_{z0} \\ \partial Y_D/\partial V_{x0} & \partial Y_D/\partial V_{y0} & \partial Y_D/\partial V_{z0} \\ \partial Z_D/\partial V_{x0} & \partial Z_D/\partial V_{y0} & \partial Z_D/\partial V_{z0} \end{bmatrix}$$

Values of the elements in the $B(t_D,t_0)$ correction matrix are found analytically from the partial derivatives of the two-body Eqs. (4) and (5) evaluated at time $t_D$ and with the six orbital elements $\alpha_i$ from Eq. (21) also found for time $t_D$. Note that both the $B(t_D,t_0)$ matrix and $\delta\mathbf{R}_D$ target position error vector are determined using the analysis developed to represent restricted three body motion. Now the velocity correction $\delta\mathbf{V}_0$ results such that when added to $\mathbf{V}_0$ the proceeding three body trajectory reaches the target position $\mathbf{R}_{DT}$ at time $t_D$. If the position error $\delta\mathbf{R}_D$ is large, the linearity implied in $B(t_D,t_0)$ prevents the determination of the exact correction. However, this difficulty is remedied by applying the calculated correction and determining a new three-body position error which is used in determining a second velocity correction to be added to the first. This may be repeated until the resulting target errors converge to zero. Convergence is usually quite rapid, which for the tests considered in this paper appeared within two or three iterations.

A linear two-body guidance scheme could be developed in a similar way. However, for this case both the $B(t_D,t_0)$ correction matrix and $\delta\mathbf{R}_D$ target position error vector are determined using a two-body trajectory mathematical model represented solely by Eqs. (4) and (5). Note that this is equivalent to solving Lambert's problem in two-body motion. This scheme is used in quantitatively judging the advantages of the analytical three-body analysis developed in this paper. A similar type of two-body guidance scheme is discussed in Ref. 3.

## Discussion of Test Cases

To demonstrate the applicability of the three-body guidance technique, a computer program was used to perform the required computations for an Earth Mars mission, and results were compared to a high-accuracy numerical integration of restricted three-body trajectories as a standard for comparison.

A 244-day Mars mission is used which starts on Julian date 2438712.5. A baseline trajectory is determined based on a six-gravitational-body numerical integration from a many-body trajectory generation program. At 140 days after launch a position error of about 30,000 km off the desired trajectory is assumed for the space vehicle. This position error causes a resulting position deviation from the desired target point near Mars at the desired time of arrival. Two target points are considered. One is on the order of $10^6$ km from Mars, and the other is ~50,000 km from Mars. Tests 1 and 2 are concerned with determining guidance corrections to be applied 140 days after launch. Test 3 evaluates a second guidance correction computed 60 days after the correction from test 1 is applied, or 200 days from launch. Two finite-mass bodies used in the restricted three-body model in each of these cases are Mars ($K_M = 42977.8$ km$^3$/sec$^2$) and the sun ($K_S = 132715440000$ km$^3$/sec$^2$). Ephemeris data for Mars was utilized in determining the various guidance corrections for the Julian date time span of the mission. Initial conditions and results of these three test cases appear in Table 1.

In test 1 the time of flight from time of correction 140 days from launch to desired time of arrival at the target point is 100 days. Prior to applying the determined correction the actual position error at the time of arrival is 166290 km, where position error is defined as target position minus three body integrated vehicle position. Following application of the three-body-determined correction the error is reduced to 2670 km using a third-order polynomial in the fitting operation referred to in Eq. (20). Using a sixth-order polynomial the final position error may be further reduced to 850 km and using an eighth-order polynomial the final position error is reduced to 413 km. Here the desired target position is on the order of 50,000 km from Mars. A correspondingly determined two-body velocity correction reduces the final position error less effectively to 5376 km. As explained above the velocity determination is iterated for large position errors until an acceptable tolerance on this error is obtained. Figure 3 indicates the convergence of the three-body computed velocity correction scheme for test 1. Note that the convergence is quite rapid and in essentially three iterations an acceptable tolerance on predicted position error is reached. Three iterations were the maximum needed in any of the tests and in fact most of them required only one or two.

In test 2 the desired target point is about 40 days outside the sphere of influence of Mars, or ~$10^6$ km from Mars. Thus, the flight time from desired time of correction is 60 days, and the actual position error at the arrival time is 22800 km before correction. Using a third-order polynomial in Eq. (20), application of the determined three-body velocity correction reduces the above final position error to 49 km. A two-body correction here reduces the error to 559 km.

Following application of the most accurate three-body guidance correction determined in test 1, a second correction was determined at 200 days from launch or 60 days after the

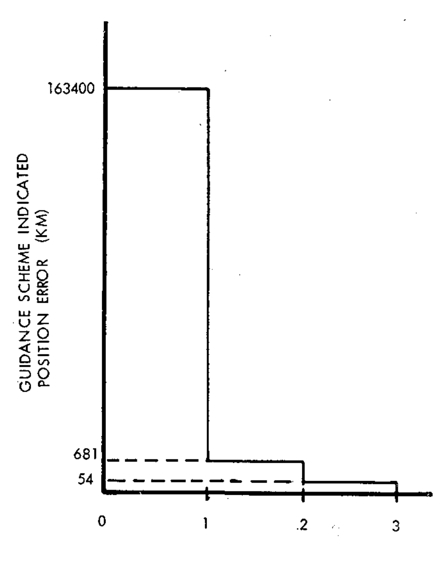


**Fig. 3 Convergence of velocity correction scheme for test 1.**

**Table 1 Comparison of two-body and three-body linear guidance scheme accuracies on an Earth Mars mission**

| | Test 1 | | Test 2 | | Test 3 | |
|---|---|---|---|---|---|---|
| | 3-body[a] | 2-body | 3-body | 2-body | 3-body[a] | 2-body |
| Errors before correction | | | | | | |
| $\epsilon_x$, km | 125,560 | | 16,127 | | −5,694 | |
| $\epsilon_y$, km | 108,975 | | 1,549 | | −4,254 | |
| $\epsilon_z$, km | −3,414 | | −4,446 | | 124 | |
| $\epsilon_R$, km | 166,290 | | 16,800 | | 7,110 | |
| $(t_D - t_0)$, days | 100 | | 60 | | 40 | |
| Velocity corrections | | | | | | |
| $\delta_x$, m/sec | 10.11 | 9.68 | 2.77 | 2.68 | −1.38 | −1.76 |
| $\delta_y$, m/sec | 11.89 | 11.81 | 0.20 | 0.18 | −0.98 | −1.15 |
| $\delta_z$, m/sec | −0.39 | −0.34 | −0.91 | −0.90 | 0.03 | 0.09 |
| $\delta_{TOT}$, m/sec | 15.60 | 15.27 | 2.92 | 2.83 | 1.69 | 2.11 |
| Errors after correction | | | | | | |
| $\epsilon_x$, km | 380 | 5,164 | 41 | 537 | −115 | 3,493 |
| $\epsilon_y$, km | 161 | 1,439 | 28 | 153 | 40 | 972 |
| $\epsilon_z$, km | −14 | −407 | −1 | −35 | 17 | −300 |
| $\epsilon_R$, km | 413 | 5376 | 49 | 559 | 122 | 3,638 |

[a] Results for an eighth-order polynomial used in Eq. (20). All position errors are based on a numerically integrated three-body trajectory as a standard for comparison.

first correction. Test 3 utilized an eighth-order polynomial in Eq. (20) for this second correction and was able to reduce a final position error at the same target point of test 1 of 7110 km down to 122 km. A two-body-determined correction could only reduce the error to 3638 km. Note that the initial conditions for this test case were obtained from six gravitational body numerical integration following application of the guidance correction as indicated above for test 1.

## Conclusions

The purpose of the test cases considered above was to indicate accuracy differences between two-body and three-body guidance schemes for interplanetary flight, utilizing a nearly exact three-body trajectory determination as a standard for determining errors. In determining total errors considering all significant perturbing effects all gravitational bodies would have to be taken into account. However, the added error due to these other effects would equally effect both correction techniques. Thus, a prerequisite for application of the techniques developed in this paper to a real mission is the accuracy to which the three-body model approximates the true many body situation. For the test cases investigated in this paper the effects of other bodies such as Jupiter create destination position errors during some phases of the mission which are fairly large when assuming the three-body model. Use of this scheme for midcourse guidance yields the best accuracies for target points in the proximity of the sphere of influence of the target planet. Increasing final position error results as the target point is moved closer to the target planet, since higher than first order terms in the series approximations made in Eqs. (14) have been neglected. An outstanding strong point of the three-body guidance correction scheme as a possible backup mode is its flexibility. It is an explicit technique independent of a precomputed reference trajectory and independent of numerical integration. The lack of dependence on numerical integration yields cost saving in terms of required computer software requirements.

Application of the basic analysis techniques discussed in this paper to a four or five gravitating body model could provide even wider applicability of this explicit guidance to manned space missions.

## Appendix A

The terms utilized in Eqs. (4) and (5) are defined as follows:

$$F = 1 - Cu^2/r_0 \tag{A1}$$

$$G = t - t_0 - Su^3/K_s^{1/2} \tag{A2}$$

$$F_1 = [\beta_0 u^3 S - u]K_s^{1/2}/rr_0 \tag{A2}$$

$$G_1 = 1 - Cu^2/r \tag{A4}$$

where

$$\beta_0 = 2/r_0 - V_0/K_s,\ r_0 = (X_0^2 + Y_0^2 + Z_0^2)^{1/2}$$

$$V_0 = (V_{x0}^2 + V_{y0}^2 + V_{z0}^2)^{1/2}$$

$$r = r_0 + (1 - r_0\beta_0)u^2C + (u - \beta_0 u^3 S)\mathbf{R}_0 \cdot \mathbf{V}_0/K_s^{1/2}$$

$$K_s^{1/2}(t - t_0) = u^2 C\mathbf{R}_0 \cdot \mathbf{V}_0/K_s^{1/2} + (1 - r_0\beta_0)u^3 S + r_0 u$$

and $C$ and $S$ are functions similar to sine and cosine functions and are defined by

$$C = C(\beta_0 u^2) = 1/2! - \beta_0 u^2/4! + (\beta_0 u^2)^2/6! - \ldots \tag{A5}$$

$$S = S(\beta_0 u^2) = 1/3! - \beta_0 u^2/5! + (\beta_0 u^2)^2/7! - \ldots \tag{A6}$$

Partial differentiation of Eq. (4) with respect to the six orbital initial condition parameters, $\alpha_1, \alpha_2, \ldots, \alpha_6$, yields the following three sets of equations:

$$\begin{gathered} X'_1 = F'_1 X_0 + G'_1 V_{x0} + F,\ X'_4 = F'_4 X_0 + G'_4 V_{x0} + G \\ X'_i = F'_i X_0 + G'_i V_{x0},\ i = 2, 3, 5, 6 \end{gathered} \tag{A7}$$

for

$$\begin{gathered} Y'_2 = F'_2 Y_0 + G'_2 V_{y0} + F,\ Y'_5 = F'_5 Y_0 + G'_5 V_{y0} + G \\ Y'_i = F'_i Y_0 + G'_i V'_{y0},\ i = 1,3,4,6 \end{gathered} \tag{A8}$$

for

$$\begin{gathered} Z'_3 = F'_3 Z_0 + G'_3 V_{z0} + F \\ Z'_6 = F'_6 Z_0 + G'_6 V_{z0} + G \\ Z'_i = F'_i Z_0 + G'_i V_{z0},\ i = 1,2,4,5 \end{gathered} \tag{A9}$$

where $X'_1 = \partial X/\partial\alpha_1$, $F'_1 = \partial F/\partial\alpha_1$, etc.; $X$, $Y$, and $Z$ are coordinates of the position vector $\mathbf{R}$; and $V_x$, $V_y$ and $V_z$ are coordinates of the velocity vector $\mathbf{V}$.

The required partial derivatives of $F$ and $G$ may be summarized as

$$\begin{bmatrix} F'_1 \\ F'_2 \\ F'_3 \end{bmatrix} = \frac{E + \dfrac{MJ}{A} - \dfrac{2}{r_0^2}\left(\mathrm{N} + \dfrac{MB}{A}\right)}{r_0}[\mathbf{R}_0] + \frac{MD}{A}[\mathbf{V}_0]$$

$$\begin{bmatrix} F'_4 \\ F'_5 \\ F'_6 \end{bmatrix} = \frac{MD}{A}[\mathbf{R}_0] - \frac{2(N + MB/A)}{K_s}[\mathbf{V}_0]$$

$$\begin{bmatrix} G'_1 \\ G'_2 \\ G'_3 \end{bmatrix} = \frac{\dfrac{HJ}{A} - \dfrac{2(Q + HB/A)}{r_0^2}}{r_0}[\mathbf{R}_0] + \frac{HD}{A}[\mathbf{V}_0]$$

$$\begin{bmatrix} G'_4 \\ G'_5 \\ G'_6 \end{bmatrix} = \frac{HD}{A}[\mathbf{R}_0] - \frac{2(Q + HB/A)}{K_s}[\mathbf{V}_0]$$

where

$$A = (u - \beta_0 u^3 S)\mathbf{R}_0 \cdot \mathbf{V}_0/K_s^{1/2} + (1 - r_0\beta_0)u^2 C + r_0$$

$$B = 0.5\, r_0 u^3(C - S) - u^4 C' \mathbf{R}_0 \cdot \mathbf{V}_0/K_s^{1/2} - u^5 S'$$

$$D = -u^2 C/K_s^{1/2},\ E = u^2 C/r_0^2$$

$$H = -r_0^2 E/K_s^{1/2},\ J = \beta_0 u^3 S - u$$

$$M = J/r_0,\ N = -u^4 C'/r_0,\ Q = -u^5 S'/K_s^{1/2}$$

$$[\mathbf{R}_0] = \begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \end{bmatrix}, \quad [\mathbf{V}_0] = \begin{bmatrix} V_{x0} \\ V_{y0} \\ V_{z0} \end{bmatrix}$$

and where derivatives of the series $C$ and $S$ given by Eqs. (A5) and (A6) are derived in Ref. 5 and may be expressed as

$$C' = (1 - \beta_0 u^2 S - 2C)/2\beta_0 u^2,\ S' = (C - 3S)/2\beta_0 u^2$$

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# Use of Jupiter's Moons for Gravity Assist


Richard W. Longman*
*Bell Telephone Laboratories, Whippany, N. J.*

AND

Alan M. Schneider†
*University of California, San Diego, La Jolla, Calif.*



**In recent years, many studies of interplanetary trajectories have described the use of a close approach to an intermediate planet to obtain savings in fuel or time in transfers to a target planet. A logical extension of this technique is the use of the target planet's moon(s) to save fuel upon arrival at the planet. This paper investigates the use of the four large moons of Jupiter to effect a transfer of a spacecraft from a hyperbolic approach orbit to an elliptic orbit about Jupiter. It is found that, for typical high-thrust trajectories from earth, at most approximately half the necessary energy change at Jupiter can be accomplished by a moon flyby. If the incoming energy is reduced somewhat using a low thrust trajectory then the flyby is sufficient to effect capture. The sensitivity of the energy change to timing and aiming errors for one-moon encounters is investigated, and several two-moon encounters are considered. The formulas derived and the techniques used are sufficiently general that they could also be applied to similar investigations of the use of the moons of other planets, such as those of Saturn and Neptune.**


## Nomenclature

$A$ = the angle between the spacecraft's velocity vector with respect to Jupiter as it approaches the moon and the moon's velocity vector with respect to Jupiter

$A'$ = the angle between the spacecraft's velocity with respect to Jupiter as it departs from the moon and the moon's velocity with respect to Jupiter


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* Member of the Technical Staff; previously Consultant, Engineering Sciences Department, The Rand Corporation, Santa Monica, Calif.

† Professor of Aerospace Engineering. Associate Fellow AIAA.